9 Curved Beams

9.1 Definition

9.2 Differences Between Bending Behavior of a Straight and a Curved Beam

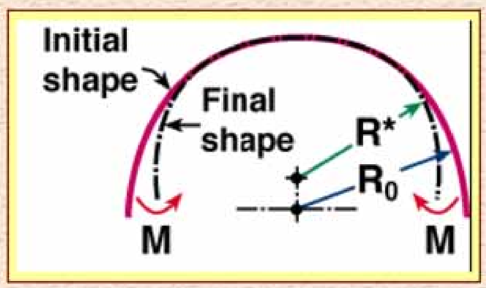
9.3 Pure Bending of Planar Curved Beams - Winkler Theory

9.4 Displacements of Curved Beams

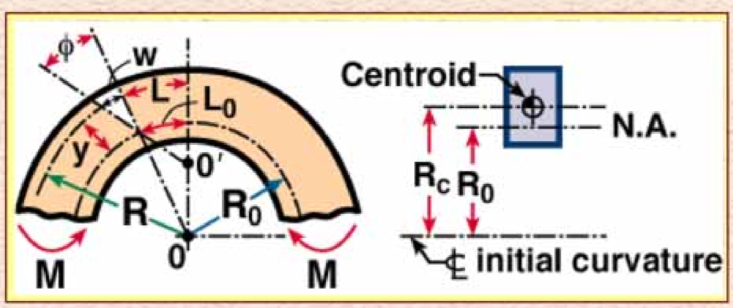
9.5 Examples

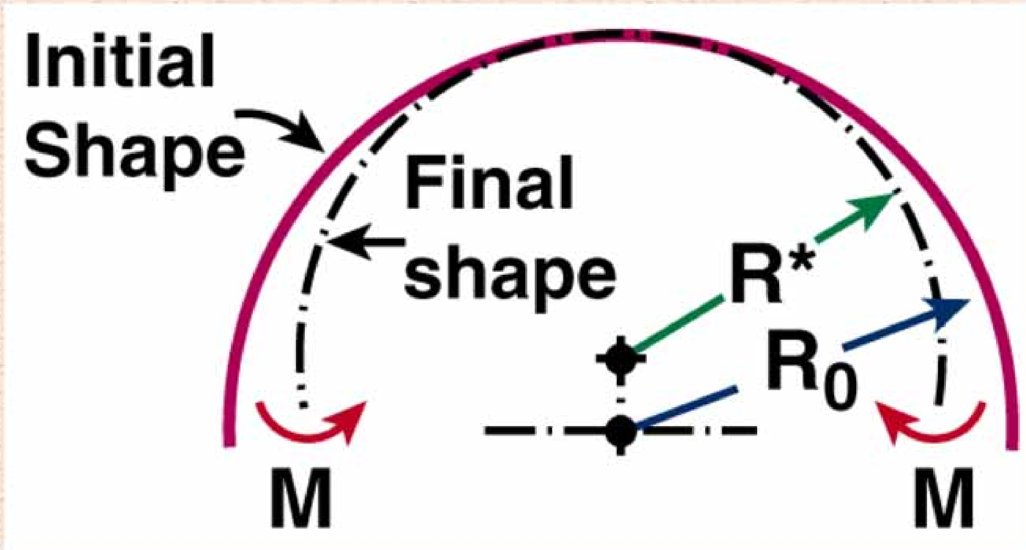
Definition

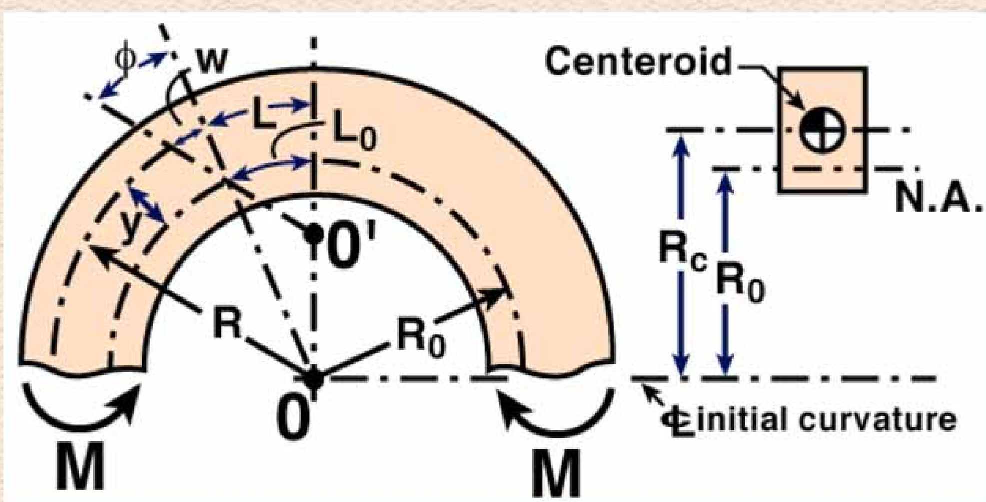
- A curved beam is a structural element for which the locus of the centroids of the cross sections is a curved line.



- When the radius of curvature is less than five times the cross sectional depth, the assumption of linear bending strain normal to the neutral axis becomes inaccurate. However, the assumption of plane cross sections remain plane after bending is still valid.




Initial
Shape
Final
shape
R*
R0
M
M


φ
w
L
L0
y
0'
R
R0
0
M
M
Centeroid
N.A.
Rc
R0
initial curvature

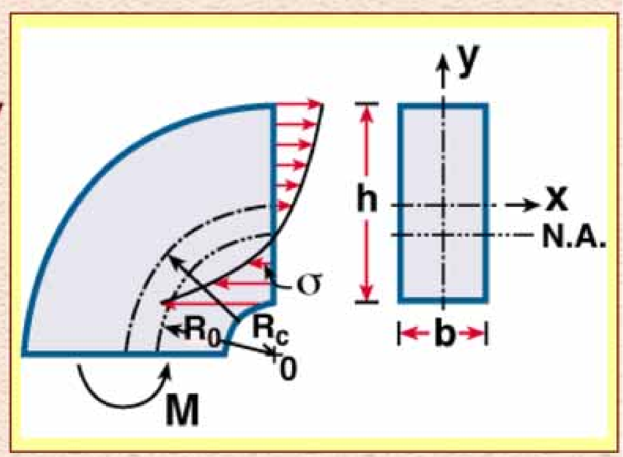
Differences Between Bending Behavior of a Straight and a Curved Beam

The assumption of plane cross sections remain plane results in linear strain distribution in straight beams, but not in curved beams.

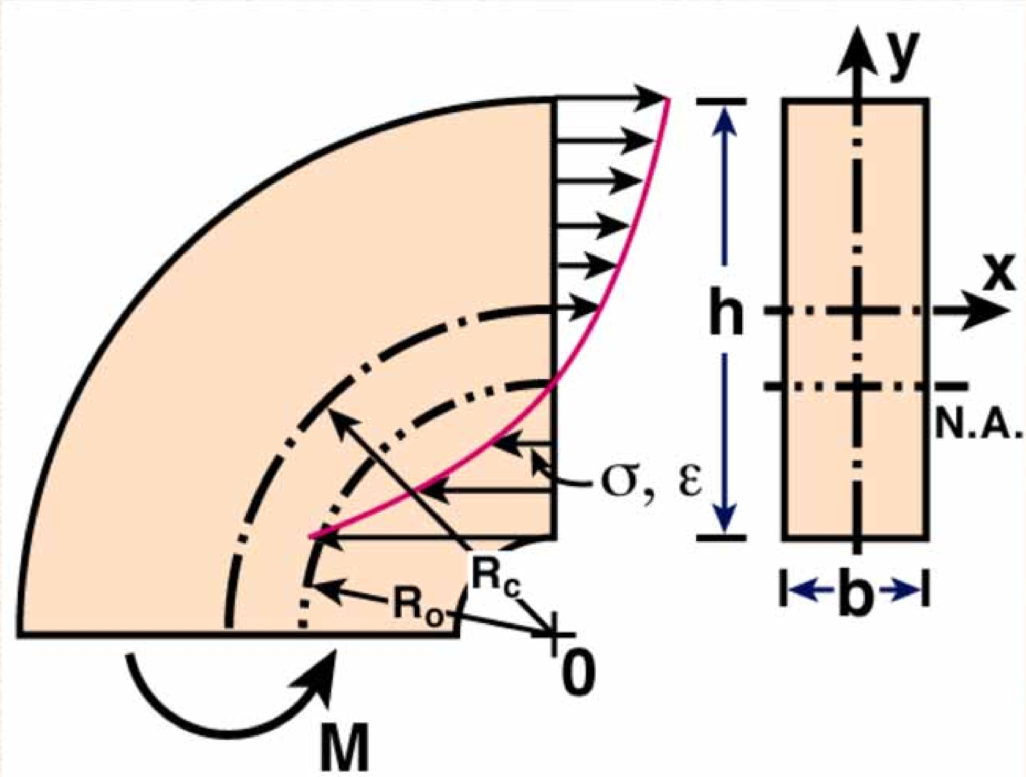
Pure Bending of Planar Curved Beams Winkler Theory

Basic Assumptions

- All cross sections possess a vertical axis of symmetry lying in the plane of the centroidal axis.
- The plane of bending coincides with the plane of symmetry of the beam.



- Plane cross sections before deformation remain plane after deformation.

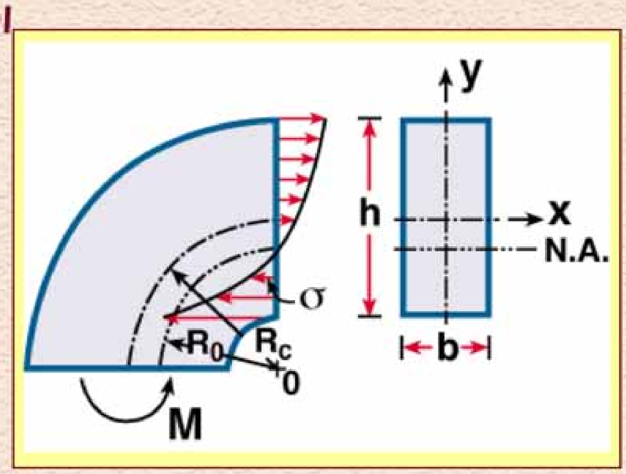


Pure Bending of Planar Curved Beams Winkler Theory

Kinematic Relations

R_0 = initial distance from the center of curvature to neutral axis (before deformation)

$\frac{1}{R_0}, \frac{1}{R^*}$ = initial and final curvatures



κ = curvature change

$$= \frac{1}{R^*} - \frac{1}{R_0}$$

ϕ = rotation of a typical cross section

$= L_0 \kappa$

Pure Bending of Planar Curved Beams Winkler Theory

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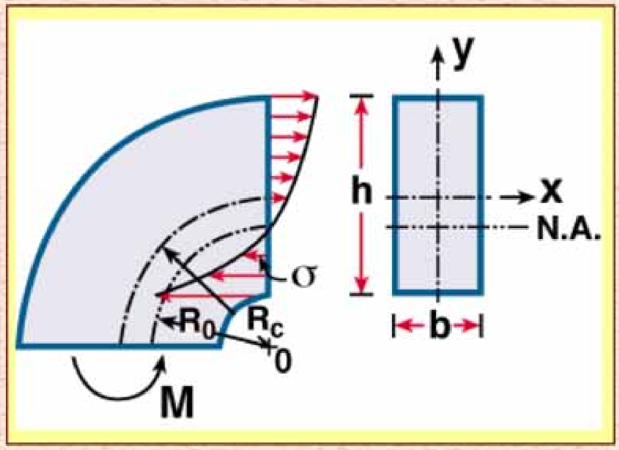


L_0 = distance measured along the neutral surface (as yet unknown)

w = axial displacement of a typical point distance y from the neutral axis

$$= y\, L_0 \kappa$$

Pure Bending of Planar Curved Beams Winkler Theory

L_0 = distance measured along the neutral surface (as yet unknown)

w = axial displacement of a typical point distance y from the neutral axis

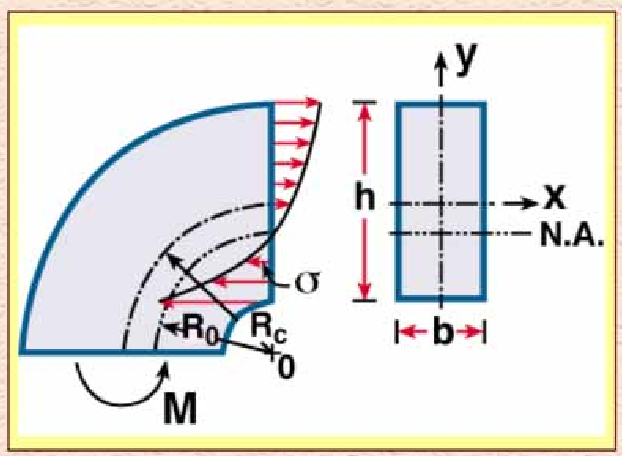
$$= y\, L_0 \kappa$$

ε = axial strain

$$= y \frac{L_0}{L} \kappa$$

but

$$\frac{L_0}{R_0} = \frac{L}{R_0 + y}$$



Pure Bending of Planar Curved Beams Winkler Theory

ε = *axial strain*

$$= y\,\frac{L_0}{L}\,\kappa$$

but

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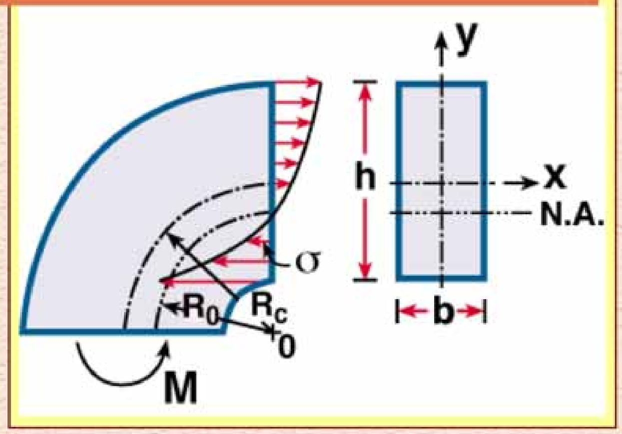


where L = distance measured along a fiber distance y from the neutral axis

Therefore, $$\varepsilon = y\,\frac{1}{1+\dfrac{y}{R_0}}\,\kappa$$

i.e., the strain distribution in the y direction is hyperbolic.

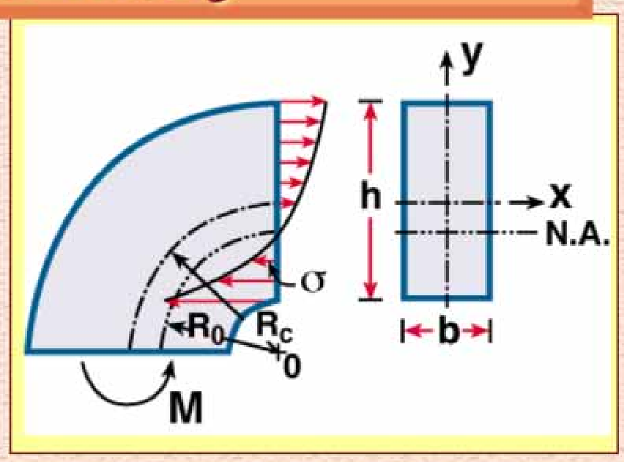
Pure Bending of Planar Curved Beams Winkler Theory

<u>Static Relations</u>

$$N = \int_A \sigma\, dA = 0$$

$$M_x = \int_A \sigma\, y\, dA$$

$$= M$$

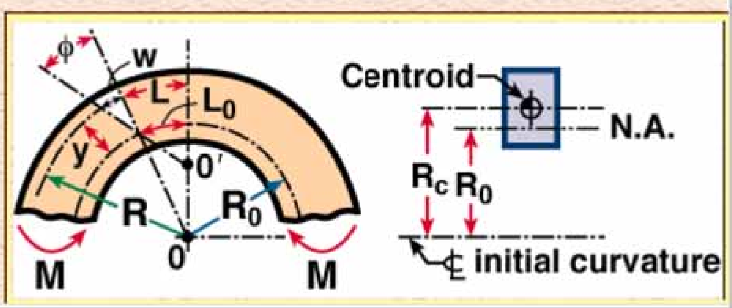


<u>Constitutive Relations</u>

for linearly elastic material

$$\sigma = E\varepsilon$$

$$\sigma\Big|_y = E\,\frac{y}{1+\dfrac{y}{R_0}}\,\kappa$$



Pure Bending of Planar Curved Beams
Winkler Theory

Constitutive Relations

for linearly elastic material

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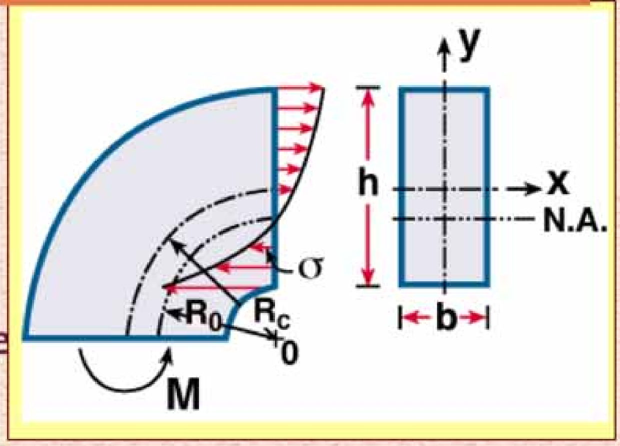
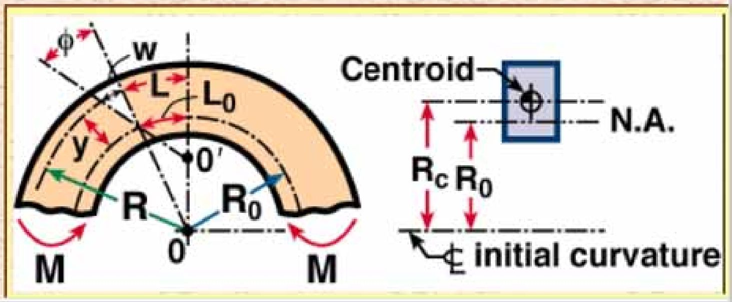


This gives the location of the neutral axis, replacing y by its expression in terms of R and R_0. $y = R - R_0$



Pure Bending of Planar Curved Beams
Winkler Theory

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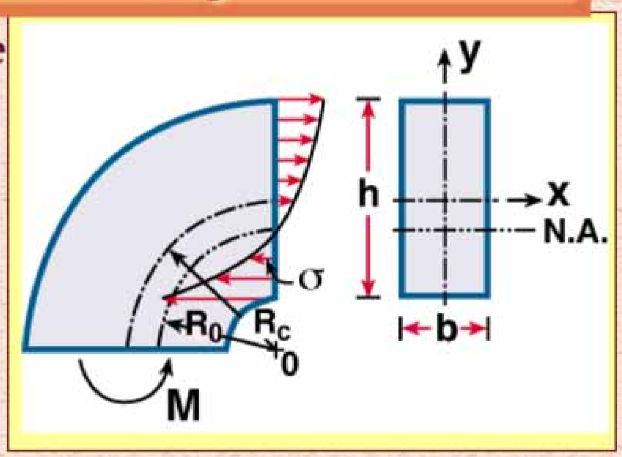
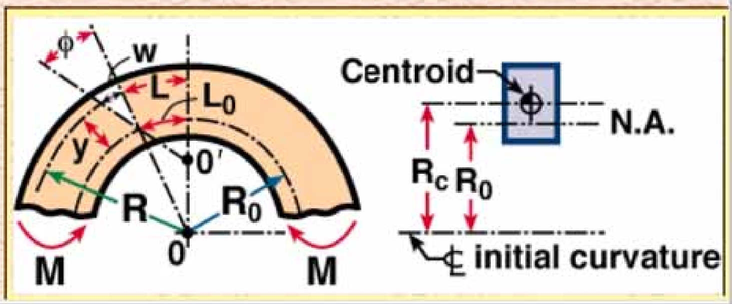
$$y = R - R_0$$



$$\int_A dA - R_0 \int_A \frac{dA}{R} = 0$$

or,

$$R_0 = \frac{A}{\int_A \frac{dA}{R}}$$



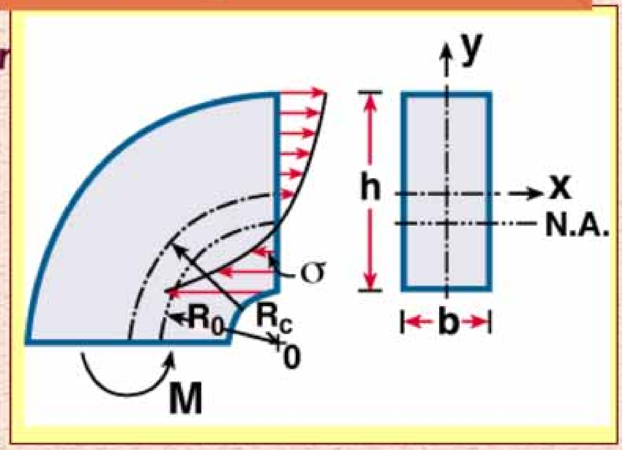
Pure Bending of Planar Curved Beams Winkler Theory

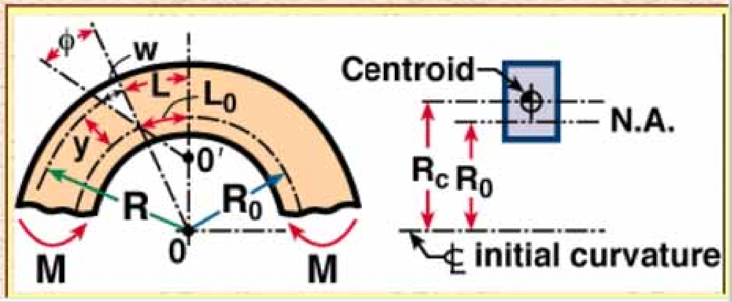
The neutral axis lies between the centoidal axis and the center of curvature.

The quantity $\int_A \frac{dA}{R}$ is a property of the cross sectional area

where $\bar{y}$ = distance from the neutral axis to the centroidal axis

$$\sigma = \frac{M_x\, y}{A\, \bar{y}\, (R_0 + y)}$$





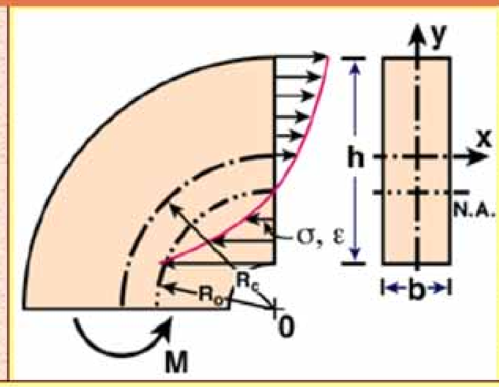
Pure Bending of Planar Curved Beams Winkler Theory

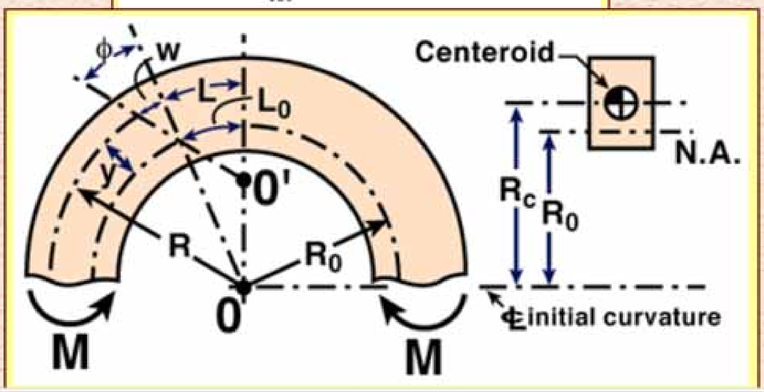
The stress distribution across the depth of the beam is hyperbolic. The maximum stress occurs at the outer fibers on the concave side of the beam.

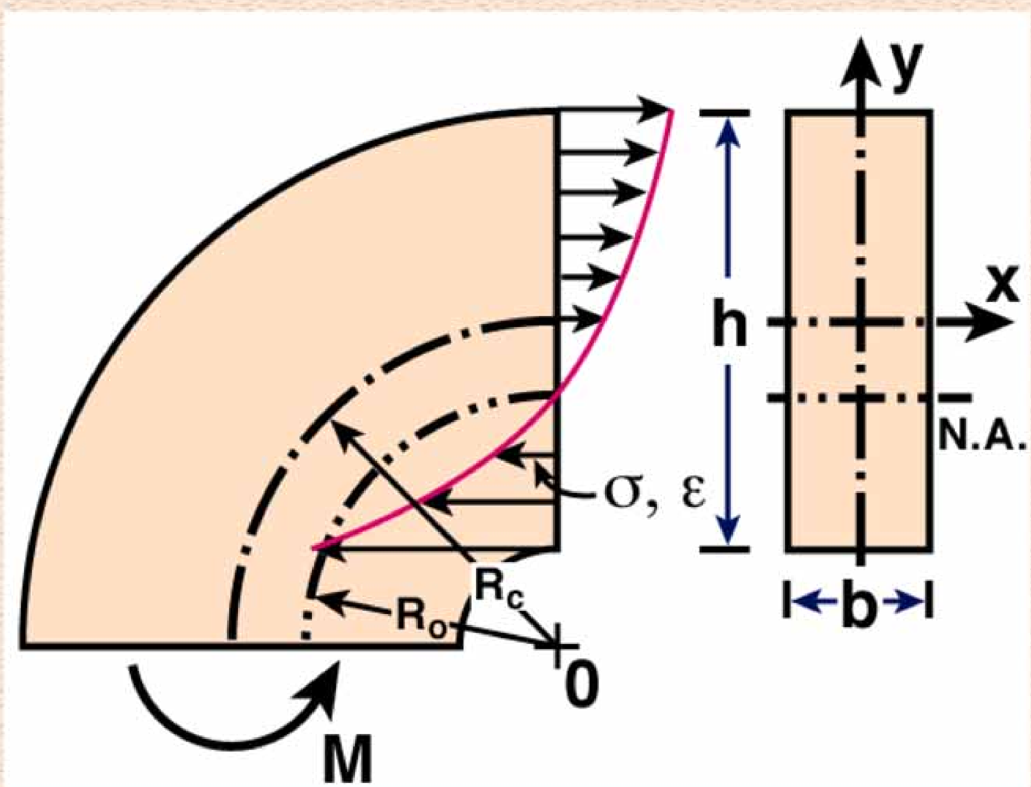
Case of Combined Axial Force and Bending Moment

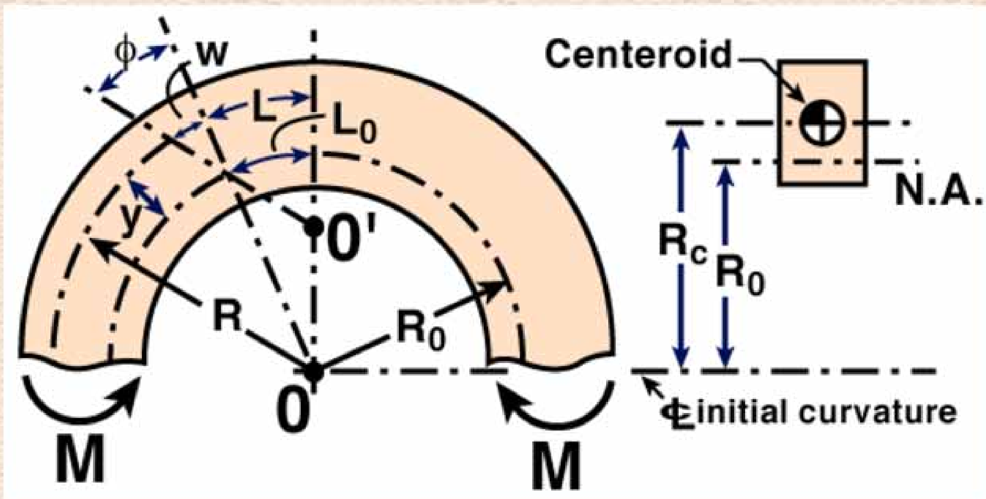
$$\sigma = \frac{N}{A} + \frac{M_x\, y}{A\, \bar{y}\, (R_0 + y)}$$

Displacements of Curved Beams





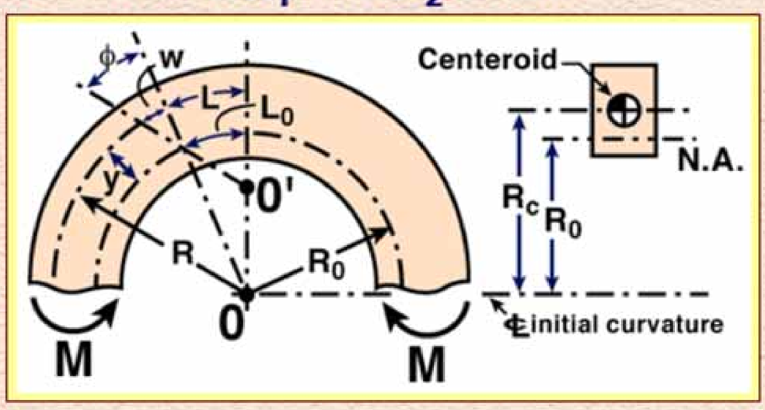




Displacements of Curved Beams

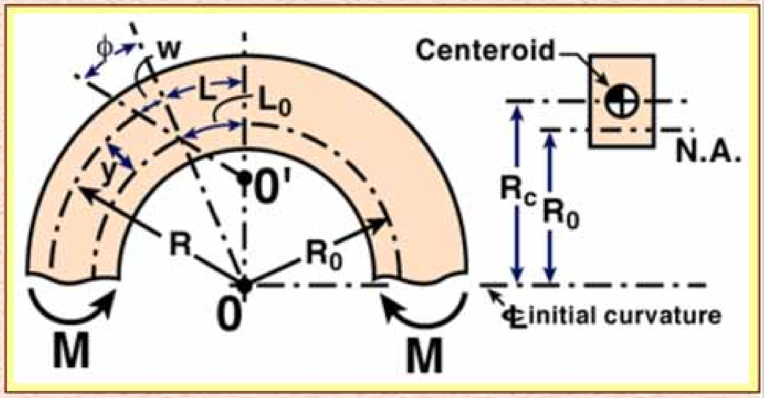
For a beam with a rectangular cross section, if the inside and outside radii are R_1 and R_2, then

$$R_0 = \frac{A}{\int_A \frac{dA}{R}} = \frac{bh}{b\int_{R_1}^{R_2} \frac{dR}{R}} = \frac{h}{\ell n\, R_2 - \ell n\, R_1}$$



Displacements of Curved Beams

$$= \frac{bh}{b\int_{R_1}^{R_2} \frac{dR}{R}}$$



$$= \frac{h}{\ell n\, R_2 - \ell n\, R_1}$$

$$= \frac{h}{\ell n\left(R_2/R_1\right)}$$

Displacements of Curved Beams

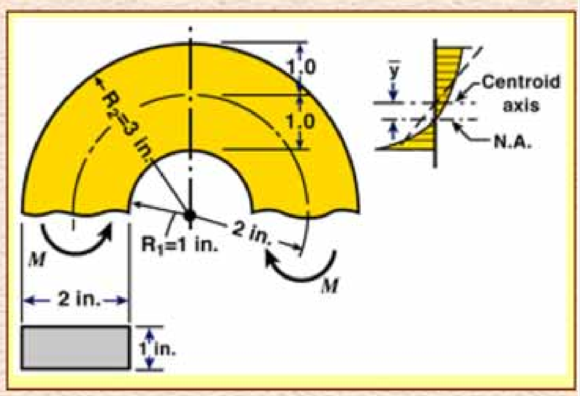
Comparison between maximum bending stresses in curved and straight beams

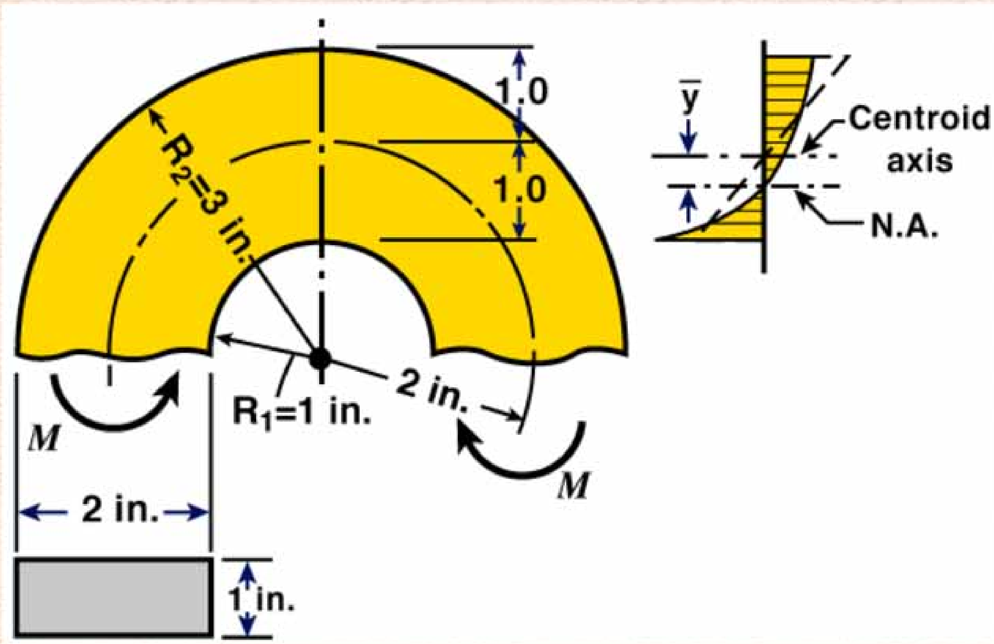
$$A = 2$$

$$R_0 = \frac{2}{\ell n\, \frac{3}{1}}$$

$$= 1.82$$

$$\bar{y} = 0.18$$





Displacements of Curved Beams

Maximum stresses occur on the inside surface

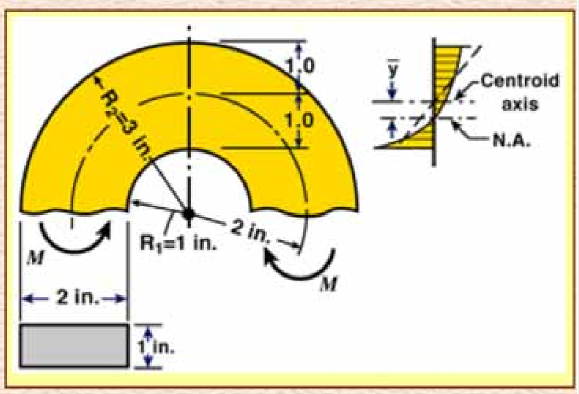
$$\sigma_{max} = \frac{M_x\,0.82}{2 \times 0.18 \times 1}$$

$$= 2.28\ M$$

For a straight beam

$$\sigma_{max} = 6\,\frac{M}{bh^2}$$

$$= 1.5\ M$$



Displacements of Curved Beams

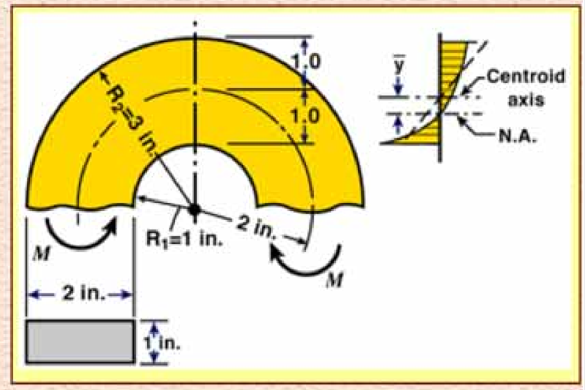
For a straight beam

$$\sigma_{max} = 6\,\frac{M}{bh^2}$$

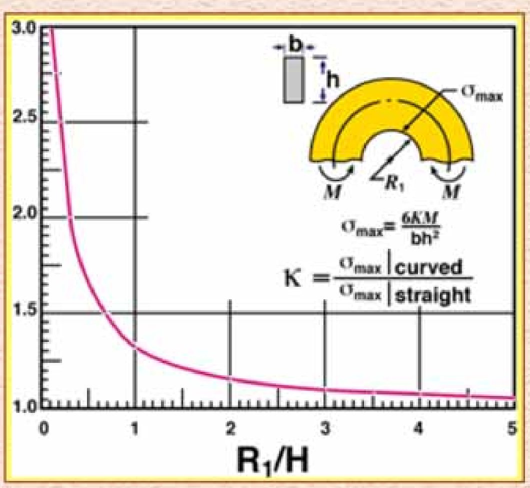
$$= 1.5\ M$$

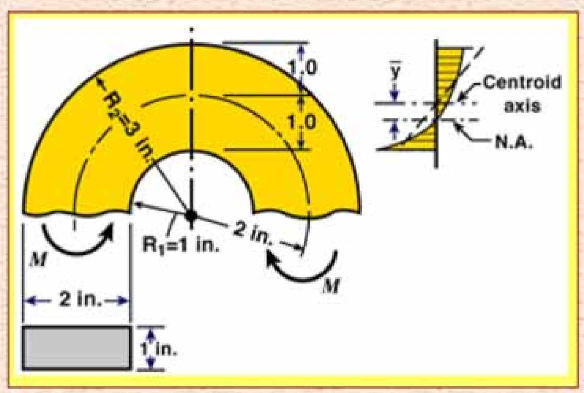
$$k = \frac{\sigma_{max}\big|_{curved}}{\sigma_{max}\big|_{straight}}$$

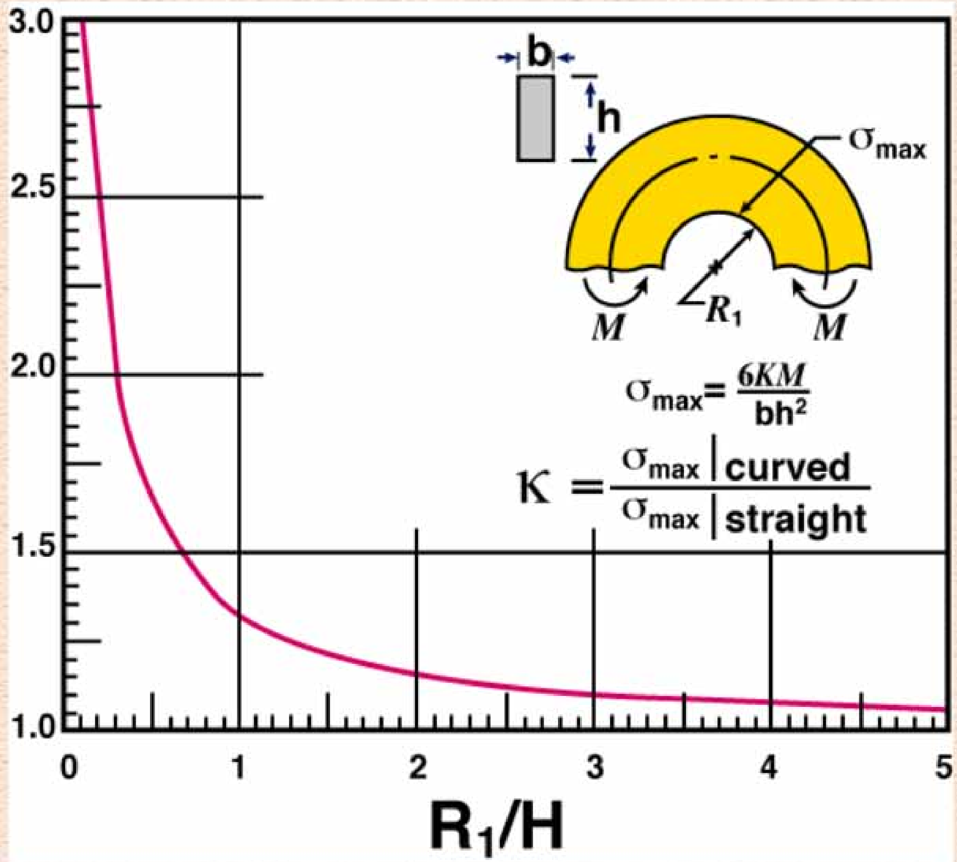
$$= 1.52$$



Displacements of Curved Beams







3.0
2.5
2.0
1.5
1.0
0
1
2
3
4
5
R1/H
b
h
σmax
R1
M
M
σmax = 6KM / bh²
K = σmax | curved / σmax | straight

Examples